Code AV-3027

B. Tech. (I[st] Year) Course A

2015-16

EMATBS-02

Engg. Mathematics - I

Note There may be Alternate Solutions to each problem. It will be decided as per merit.

Section A

(1) (i) Rolle's Theo: If a function defined on $[a,b]$ is such that (i) it is continuous in $[a,b]$ (ii) is derivable in (a,b) (iii) $f(a)=f(b)$, then there exists at least one point c $(a<c<b)$ such that $f'(c)=0$

(ii) If $y = e^{ax}\cos(bx+c)$, then

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

(iii) $$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(iv) $$\lim_{x\to 0}\frac{\log x}{\cot x} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = -\lim_{x\to 0}\frac{\sin^2 x}{x} \quad \left(\frac{0}{0}\right)$$

$$= -\lim_{x\to 0}\frac{2\sin x\cos x}{1} = 0$$

(v) If $I_n = \int \cot^n x\,dx = \int \cot^{n-2}x \cdot \cot^2 x\,dx = \int \cot^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$

$$= \int \cos^{n-2}x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2}x\,dx$$

$$I_n = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$$

(vi) If $I_n = \int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

Integrating by parts, we get

$$I_n = \sec^{n-2}x \cdot \tan x - \int\{(n-2)\sec^{n-3}x \cdot \sec x\tan x\}\tan x\,dx$$

Thus $$I_n = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

... t^{n-1} ||^{el} to y axis, equate merely a constant.

The asymptotes $\|^{el}$ to y axis are $x = 0$ and $x = 1$

(viii) $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases}$

or

(ix) $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$

Variables are seperable, so

$$\frac{y}{\sqrt{1-y^2}}\,dy + \frac{x}{\sqrt{1-x^2}}\,dx = 0$$

On itegrating, we get

$$\sqrt{1-x^2} + \sqrt{1-y^2} = C$$

(x) Bernoulli Equation

The equation $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x, is usually called Bernoulli's Eq.

Section B

2 (a) We have

$y = (\sin^{-1} x)^2$,

Diff. w. r.t x, we get

$$y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow (1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y$$

Again differentiating we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$$

cancelling $2y_1$ throughout, we get

$$\boxed{(1-x^2)y_2 - xy_1 - 2 = 0}$$

diff. it n times by Leibnitz theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - [xy_{n+1} + n(1)y_n] = 0$$

(b) given $y = e^{2x} \cos^2 x \sin x$

$$\cos^2 x \sin x = \frac{1}{2} \cos x . \sin 2x = \frac{1}{4}(2\sin 2x \cos x) = \frac{1}{4}(\sin 3x + \sin x)$$

$$\therefore y = \frac{1}{4}\left[e^{2x}\sin 3x + e^{2x}\sin x\right]$$

Therefore

$$y_n = \frac{1}{4}\left[(13)^{\frac{n}{2}} e^{2x} \sin\left(3x + n\tan^{-1}\frac{3}{2}\right) + (5)^{\frac{n}{2}} e^{2x} \sin\left(x + n\tan^{-1}\frac{1}{2}\right)\right]$$

3 (a) We have $x^4 = 16y^2 = 16 \times 4x$

$x(x^3 - 64) = 0 \Rightarrow x = 0$ or $x = 4$

The curves intersect at point $(0, 0)$ and $(4, 4)$

For curve $x^2 = 4y \Rightarrow \frac{dy}{dx} = \frac{x}{2}$

At $(0,0)$, slope of tangent(m_1) to $x^2 = 4y$ is $m_1 = \frac{0}{2} = 0$; slope of tangent (m_2) to $y^2 = 4x$ is $\frac{2}{0} = \infty$

At $(4, 4)$, slope of tangent(m_1) to $x^2 = 4y$ is $\frac{4}{2} = 2$;

Slope of tangent to $y^2 = 4x$ is $\frac{2}{4} = \frac{1}{2}$

$\therefore$ Angle of intersection of curves

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{2 - \frac{1}{2}}{1 + 2.\frac{1}{2}} = \tan^{-1}\frac{3}{4}$$

(b) Stationary points are at $x = +1$, $x = -1$ and $x = \frac{1}{2}$

So in the interval $(0,2)$ only two points $x = \frac{1}{2}$ and $x = 1$ lies.

So $f''(x) < 0$ at $x = \frac{1}{2}$, So $[f(x)]_{x=\frac{1}{2}} = \frac{39}{16} = 2\frac{7}{16}$ (Max)

Again $f''(x) > 0$ at $x = 1$, So $[f(x)]_{x=1} = 2$ (Min)

4 (a) $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

$= \lim\limits_{x\to 0}\left(\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5}{x}\right)^{\frac{1}{x^2}}$

$= \lim\limits_{x\to 0}\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\dots\right)^{\frac{1}{x^2}}$

$= \lim\limits_{x\to 0}\left(1+tx^2\right)^{\frac{1}{x^2}}$ Put $t = \frac{1}{3}+\frac{2}{15}x^2+\dots$

$= \lim\limits_{x\to 0}\left[\left(1+tx^2\right)^{\frac{1}{tx^2}}\right]^t$ Since $\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}} = e$

$= \lim\limits_{x\to 0} e^t = \lim\limits_{x\to 0} e^{\frac{1}{3}+\frac{2}{15}x^2+\dots}$

$= e^{\frac{1}{3}}$

(b) Put $x=1$ & $y=m$, we get

$\phi_3(m) = m^3-2m^2-m+2$

So $\phi_3(m)=0 \Rightarrow m=-1, 1$ and 2

Also $c = -\frac{\phi_2(m)}{\phi_3'(m)}$ $\phi_2(m) = 3m^2-7m+2$

Thus $c=-1$, when $m=1$

$c=-2$ when $m=-1$

$c=0$ when $m=2$

Thus asymptotes are

$y=x-1$, $y=-x-2$ and $y=2x$

5 (a) Given parabola $y^2=4ax$ or

$x=at^2$ and $y=2at$.

If dashes denotes differentiation w.r.t t, we get

$x'=2at$, $y'=2a$

$x''=2a$, $y''=0$

$$\therefore \rho \text{ at } (at^2, 2at) = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''}$$

$$= 2a(1+t^2)^{3/2}$$

If $P(t_1)$ and $Q(t_2)$ be the extremities of focal chord of parabola, then

$$t_1 t_2 = -1 \text{ or } t_2 = -\frac{1}{t_1} \quad \text{—(i)}$$

$$\therefore \rho_1 \text{ at } P(t_1) = 2a(1+t_1^2)^{3/2}$$

and ρ_2 at $Q(t_2) = 2a(1+t_2^2)^{3/2}$

Thus

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}\left\{(1+t_1^2)^{-1} + (1+t_2^2)^{-1}\right\}$$

$$= (2a)^{-2/3}\left\{\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_2}\right\}$$

Putting $t_2 = -\frac{1}{t_2}$

$$\therefore (\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}$$

(b) Given $z = f(x+ct) + \phi(x-ct)$

$$\therefore \frac{\partial z}{\partial x} = f'(x+ct)\frac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial x}(x-ct)$$

$$\therefore \frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$$

Again $\frac{\partial z}{\partial t} = c\left[f'(x+ct) - \phi'(x-ct)\right]$

$$\therefore \frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \phi''(x-ct)\right]$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial t^2}$$

6 (a) Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$

$$= \left[\cos^m x \cdot \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x (-\sin x) \frac{\sin nx}{n} dx$$

$$I_{m,n} = \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \, (\sin nx \sin x) dx$$

Since $\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

$\Rightarrow \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$

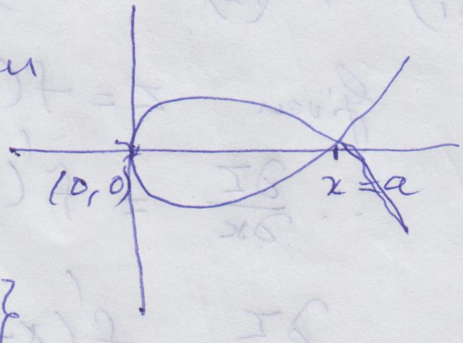
$$\therefore I_{m,n} = \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \left[\cos(n-1)x - \cos nx \cos x\right] dx$$

$$\therefore I_{m,n} = \frac{m}{n}\left[I_{m-1,n-1} - I_{m,n}\right]$$

Thus

$$I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$$

(b) The loop lies between ~~(0,0)~~ limits $x=0$ and $x=a$.



We have ~~y = √x(x−a)~~

$$y = \frac{1}{(3a)^{1/2}} \left\{x^{\frac{3}{2}} - a x^{\frac{1}{2}}\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(3a)^{\frac{1}{2}}} \left\{\frac{3}{2} x^{\frac{1}{2}} - \frac{a}{2} x^{-\frac{1}{2}}\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{3a}} \left(\frac{3x-a}{\sqrt{x}}\right)$$

$$\therefore \text{Perimeter of loop} = 2\int_{x=0}^{x=0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\int_0^a \sqrt{1 + \frac{(3x-a)^2}{12ax}} \, dx$$

$$= 2 \int^a \sqrt{9x^2 + 6ax + a^2} \quad dx$$

$= \frac{1}{\sqrt{3a}} \int_0 \frac{3x+a}{\sqrt{x}} dx = \frac{1}{\sqrt{3a}} \int_0 \left(3x^{\frac{1}{2}} + a x^{-1/2}\right) dx$

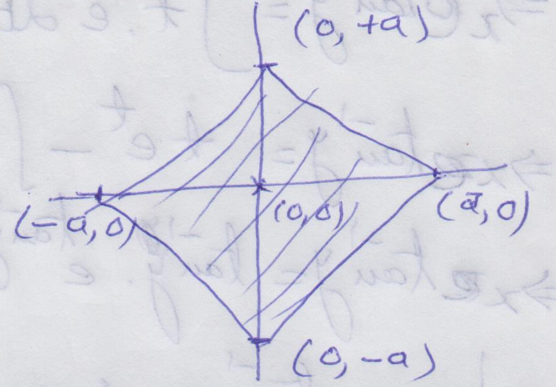
$= \frac{4a}{3}$

7 (a) The curve is

$x^{2/3} + y^{2/3} = a^{2/3}$

It's parametric form is

$x = a\cos^3\theta, \quad y = a\sin^3\theta$



The whole area is

$$A = 4\int_{x=0}^{x=a} y\,dx$$

$$= 4\int_{\theta=0}^{\theta=\pi/2} a\sin^3\theta \,.\, 3a\cos^2\theta . (-\sin\theta)\,d\theta$$

$A = 3\pi a^2$

(b) $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$

$\therefore dx = \frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \frac{1}{2}\sin^{-1/2}\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2}\frac{\sin^{-\frac{1}{2}}\theta\cos\theta}{\cos\theta}\,d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta\,d\theta = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

8 (a) This equation contains y^2 and $\tan^{-1}y$, and so it is not linear.

It can be rewritten as

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{\ldots} = \frac{\tan^{-1}y}{\ldots}$$

So $\quad I.F. = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

Solution is

$$\left(x\, e^{\tan^{-1} y}\right) = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + C$$

Put $\tan^{-1} y = t$ $\quad \therefore \frac{dy}{1+y^2} = dt$

$$\Rightarrow x\, e^{\tan^{-1} y} = \int t \cdot e^t\, dt + C$$

$$\Rightarrow x\, e^{\tan^{-1} y} = t e^t - \int 1 \cdot e^t\, dt + C$$

$$\Rightarrow x\, e^{\tan^{-1} y} = \tan^{-1} y \cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + C.$$

$$\Rightarrow \boxed{x = \tan^{-1} y - 1 + C e^{-\tan^{-1} y}}$$

(b) Put $x + y = t$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dt}{dx} - 1 = \sin t + \cos t$$

variables seperated, so

$$\int dx = \int \frac{dt}{1 + \sin t + \cos t} + C$$

Put $t = 2\theta$ $\quad \Rightarrow \theta = \frac{x+y}{2}$

$$\therefore x = \int \frac{2\, d\theta}{1 + \sin 2\theta + \cos 2\theta} + C$$

$$\Rightarrow x = \int \frac{2\, d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + C$$

$$\Rightarrow x = \int \frac{\sec^2\theta\, d\theta}{1 + \tan\theta} + C$$

$$\Rightarrow x = \log(1 + \tan\theta) + C$$

$$\Rightarrow x = \log\left(1 + \tfrac{1}{2}(x+y)\right) + C$$

9 (a) By doing stepwise, we get solution as

$$y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

(b) A.E. is $(m-2)^2 = 0$

i.e. $m = 2$, and 2

$$C.F. = (c_1 + c_2 x)e^{2x}$$

$$P.I.\ \text{of} = \frac{1}{D^2 - 4D + 4} e^x = \frac{1}{1^2 - 4 + 4} e^x = e^x$$

$$P.I.\ \text{of}\ \frac{1}{(D-2)^2} x^2$$

$$= \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 + (-2)\left(-\frac{D}{2}\right) + \frac{(-2)(-3)}{\underline{|2}}\left(-\frac{D}{2}\right)^2 + \cdots\right] x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3D^2}{4} + \cdots\right] x^2$$

$$= \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$P.I.\ \text{of} = \frac{1}{(D-2)^2}\cos x = \frac{1}{D^2 - 4D + 4}\cos x$$

$$= \frac{1}{-1^2 - 4D + 4}\cos x = \frac{1}{(3 - 4D)}\cos x$$

$$= \frac{3 + 4D}{(3-4D)(3+4D)}\cos x = \frac{(3+4D)}{9 - 16D^2}\cos x$$

$$= \frac{3\cos x - 4\sin x}{9 - 16(-1)^2} = \frac{1}{25}(3\cos x - 4\sin x)$$

Hence

$$y = (c_1 + c_2 x)e^{2x} + e^x + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$+ \frac{1}{\ }(3\cos x - 4\sin x)$$

10 (a) Wave Equation is (Alternative Solutions are there)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (1)$$

Assume the solution (1) is of form $y = X(x)\,T(t)$ where X is function of x and T is function of t only

Thus $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Putting in (1), we get

$$XT'' = X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \quad -(2)$$

Clearly l.h.s. of (2) is function of x only and r.h.s. is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Thus (2) leads to

$$\frac{d^2X}{dx^2} - kX = 0 \quad (3) \text{ and } \frac{d^2T}{dt^2} - c^2 k T = 0 \quad (4)$$

Solving (3) and (4), we get

(i) when k is positive (say p^2),

then $X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{cpt} + c_4 e^{-cpt}$

(ii) when k is negative (say $-p^2$), then

$X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$

(iii) when k is zero, then

$X = c_9 x + c_{10}$, $T = c_{11} t + c_{12}$

Thus possible solutions are

$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

$\ldots (c_7 \cos cpt + c_8 \sin cpt)$

10 (b) Let $K = c^2$, (Alternative methods may be there)

Thus equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—} ①$$

Assume the solution is of the form

$$u(x,t) = X(x).\,T(t)$$

proceeding as in 10(a), we get

$$\frac{X''}{X} = \frac{T'}{c^2 T} \quad \text{—} ②$$

Clearly the left hand side of ② is a function of x only and the right hand side is a function of t only. Then ① leads to the ordinary diff. Eq:

$$\frac{d^2 X}{dx^2} - \lambda X = 0 \quad \text{—} ③ \qquad \text{and} \qquad \frac{dT}{dt} - \lambda c^2 T = 0 \quad \text{—} ④$$

Solving ③ & ④, we get

(i) when λ is positive and (say p^2),

$$X = C_1 e^{px} + C_2 e^{-px}\,; \quad T = C_3 e^{c^2 p^2 t}$$

(ii) when λ is negative and (say $-p^2$)

$$X = C_4 \cos px + C_5 \cos px, \quad T = C_6 e^{-c^2 p^2 t}$$

(iii) When λ is zero,

$$X = C_7 x + C_8\,, \quad T = C_9$$

The only ~~a~~ physical nature problem of heat equation is

$$\boxed{U = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t}}$$

11 (a) The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{\tan(y-3x)}$$

Integrating first two get

$$y = 3x + a \Rightarrow y - 3x = a \quad \text{--- (1)}$$

Taking first and Third, we get

$$\frac{dx}{1} = \frac{dz}{\tan a}$$

$$\Rightarrow x \tan a = z + b \quad \text{--- (2)}$$

Thus (1) & (2) form system of solutions.

(b) We know

$$DD'z = e^x . e^y$$

Integrating w.r.t x, we get

$$D'z = e^x . e^y + f(y) \quad \text{--- (ii)}$$

Now integrating (ii) w.r.t y, we get

$$z = e^{x+y} + \int f(y)dy + \phi(x)$$

Thus $z = e^{x+y} + F(y) + \phi(x)$

(c) Lagrange's equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow \text{from first two}$$

fractions we write $\cot x \, dx = \cot y \, dy$

$$\Rightarrow \log \sin x = \log \sin y + \log c_1 \Rightarrow \frac{\sin x}{\sin y} = c_1$$

Similarly $\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow c_2 = \frac{\sin y}{\sin z}$

Thus $\frac{\sin x}{\quad} = \phi\left(\frac{\sin y}{\sin z}\right)$